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Holographic domains of anti-de Sitter space

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Abstract: An AdS_4 brane embedded in AdS_5 exhibits the novel feature that a four-dimensional graviton is localized near the brane, but the majority of the infinite bulk away from the brane where the warp factor diverges does not see four-dimensional gravity. A naive application of the holographic principle from the point of view of the four-dimensional observer would lead to a paradox; a global holographic mapping would require infinite entropy density. In this paper, we show that this paradox is resolved by the proper covariant formulation of the holographic principle. This is the first explicit example of a time-independent metric for which the spacelike formulation of the holographic principle is manifestly inadequate. Further confirmation of the correctness of this approach is that light-rays leaving the brane intersect at the location where we expect four-dimensional gravity to no longer dominate. We also present a simple method of locating CFT excitations dual to a particle in the bulk. We find that the holographic image on the brane moves off to infinity precisely when the particle exits the brane's holographic domain. Our analysis yields an improved understanding of the physics of the AdS_4/AdS_5 model.

Keywords: Black Holes in String Theory, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

The holographic principle [1, 2] has frequently been interpreted in a space-like sense. The space-like formulation asserts that the entropy in a spatial volume V enclosed by a surface of area A cannot exceed $A/4$, in Planck units. At each instant of time, the complete information about physics in V can be encoded on the boundary A, at a density of no more than one bit per Planck area. The space-like formulation is ambiguous as it involves an arbitrary coordinate choice. The area A and the volume V are fully specified only after a particular equal-time slice has been picked.

More significantly, in strongly time-dependent backgrounds the space-like holographic principle can actually be violated. Counter-examples were first identified in cosmological spacetimes [3]. The entropy of sufficiently large super-horizon regions will exceed their surface area. More generally, violations are easily found among systems in which gravitational expansion or contraction dominates the dynamics [4]. Inside a black hole, for example, a star undergoing gravitational collapse will have arbitrarily small surface area shortly before it is crushed in a singularity.

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Building on [3], these difficulties were resolved by a covariant reformulation [4, 5, 6]. In the covariant holographic principle, the entropy bound, $S \leq A/4$, is not modified. But instead of the entropy on a space-like hypersurface, one considers the entropy on a null hypersurface, or "light-sheet". Given an area A (any open or closed codimension 2 spatial surface), a light-sheet is generated by contracting light-rays emanating orthogonally from A.

No counter-example to the covariant entropy bound is known. What further distinguishes the light-sheet construction from previous formulations of the holographic principle is its ability to relate *any* area A uniquely to adjacent bulk regions, even if A is not closed. (This particular feature will be crucial to our construction of holographic bulk regions associated with branes.) A better understanding of the underlying reason for the success of the light-sheet formulation might well lead to new insights into the connection between classical geometry and its statistical origin in a quantum theory of gravity.

The covariant holographic principle is intriguing conceptually and has proven essential in time-dependent backgrounds. However, the covariant formulation has not yet been brought to bear on holography's most detailed manifestation, the AdS/CFT correspondence. For certain classes of asymptotically anti-de Sitter space-times, the AdS/CFT correspondence [7, 8, 9] asserts that the physics in the bulk of the space-time is fully described by a conformal field theory (CFT) on the boundary. The number of degrees of freedom of the CFT is in accordance with the holographic bound [10].

In the AdS/CFT correspondence, no contradictions to the space-like formulation had been noticed until now. At least from a global point of view, this is not surprising. Under the *space-like condition* (the existence of a light-sheet that is *complete*, i.e., has no boundaries other than A), the covariant holographic principle implies the space-like holographic principle as a special case [4]. The space-like condition is met by some[1] closed surfaces around any isolated, weakly gravitating matter system. In a perturbative treatment of AdS, the entire space satisfies the space-like condition [5]; a sphere near spatial infinity possesses a complete light-sheet.[2]

Space-like holography has also been sufficient for discussions of the Randall-Sundrum (RS) model [12]. This model describes a brane whose metric is 3+1 dimensional Minkowski space. The brane moves in a portion of AdS_5, at fixed radial position in Poincaré coordinates. From the AdS/CFT correspondence, one expects that objects in the bulk can alternatively be described by excitations of a CFT on the brane [13]. The CFT must be

[1]Note that any system can be surrounded by surfaces of arbitrarily small area, by choosing a suitably wiggled equal-time slice. (One can obtain such slices by piecing together sections of equal time surfaces of observers moving nearly at the speed of light relative to the system.) This is a somewhat unnatural, but perfectly admissible choice, and it obviously leads to a violation of the space-like holographic principle (though not of the covariant version). However, the break-down of the space-like holographic principle exposed in this paper is unrelated to this particular weakness.

[2]Space-like holography need not be challenged even when black holes are allowed in the bulk. By black hole complementarity [11], the CFT will be in one-to-one correspondence with the portion of the bulk consisting of the black hole exterior and the black hole horizons. This part of the geometry is not strongly time-dependent, and the space-like holographic principle is expected to hold.

coupled to the four-dimensional graviton localized on the brane. In section 2.1 we explain why space-like holography has not led to contradictions in this model: it implies entropy bounds that are actually weaker than those obtained from the covariant formulation.

However, from the vantage point of space-like holography, the case of an AdS_4 brane embedded in AdS_5 (KR) [14] poses a puzzle. The KR model presents a significant new development that broadens greatly the class of theories that might exhibit localization. It illustrates that the localization of gravity does not depend on certain restrictive conditions satisfied by the RS model. These include the necessity for a finite volume bulk space and a decreasing warp factor as an asymptotic boundary condition. The localization occurs due to local features of the geometry; regions far from the brane are not relevant. This is discussed in detail in [14].

In the RS model, the graviton zero mode needs to be incorporated as an explicit addition to the original CFT and to the original five-dimensional space. One remarkable property of the KR model is that from the point of view of the boundary CFT, the four-dimensional graviton is also a bound state [15, 16]. Moreover, the KR model displays features not previously seen in any four-dimensional theory of gravity. First, the graviton has a small mass, whose value is set by the curvature scale of the AdS_4.

Second, in the KR model four-dimensional gravity dominates only over a fraction of the volume. The graviton appears as a light bound state that only extends over a finite region of the bulk space. This means that observers localized to different regions of the space would see the dimensionality of the space as different.

However, the space-like holographic principle will not reflect this distinction. In fact, we show in section 2.2 that space-like holography would assign a bulk region of infinite entropy to a finite area region on the brane, clearly a contradiction.

Some clue to the resolution of this paradox is that the boundary of the bulk space is not only the AdS_4 brane, but also half of the AdS_5 boundary. One might expect different holographic regions to be associated with the different portions of the boundary. This is particularly relevant as only part of the space even possesses four-dimensional gravity, and the holographic correspondence would need to distinguish such a region.

A further question that arises in the context of the KR model is the interpretation of Newton's constant. In the RS model, G_{N} is determined by the normalization of the zero mode, which is in turn determined by the volume of the bulk space. By contrast, the volume of the bulk in the KR model is infinite and clearly does not correspond to the graviton normalization.

In the main part of our paper, we show that the resolution of these difficulties lies in the covariant formulation of the holographic principle. This is important in that it is the first time-independent space for which this distinction is essential. We will also see evidence that the covariant treatment gives the correct description of the physics of the AdS_4/AdS_5 system, increasing our confidence both in the consistency of the KR model and in the covariant formulation itself.

Furthermore, we find the four-dimensional region is essentially equivalent to the holographic domain assigned to the brane. This is also the region whose volume yields the proper graviton normalization and therefore G_{N}.

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In section 3, we review the covariant entropy bound and propose its application to branes. The construction of light-sheets starting at a given area is described in detail in section 3.1. Light-sheets must be terminated at caustics, when the generating light-rays intersections In section 3.2, we explain how to locate caustics, and we show how this calculation simplifies under certain conditions. A brane is not an area but a time-like hypersurface; it contains a whole sequence of areas, giving rise to a sequence of light-sheets. Section 3.3 describes our adaption of the covariant entropy bound to this case. After resolving a slicing ambiguity, we define the holographic domain, $\mathcal{D}$, of a given brane as the bulk region covered by the sequence of light-sheets. This is the bulk region whose physics we may expect to be dual to holographic states on the brane.

In section 4, we apply the covariant formalism to an AdS_4 brane in AdS_5. We determine the caustics of light-sheets that start on the brane, and we find that they occur at a finite distance from the brane. Thus we demonstrate that the holographic domain associated with an AdS_4 brane does not include the entire AdS_5 bulk, but extends only from the brane to another AdS_4 hypersurface near the minimum of the warp factor. The remaining portion of the bulk constitutes half of the AdS_5 space-time. The physics in this portion cannot be holographically represented on the brane.

This raises an important question. Consider a bulk particle moving away from the brane. When the particle crosses the minimum of the warp factor, it leaves the holographic domain of the brane and should no longer have a CFT dual on the brane. But how can the CFT excitation dual to the particle actually disappear from the brane?

In section 5, we develop an extremely simple prescription relating the position of a bulk excitation to the locus of the dual CFT excitation on a brane. We use only causality, employing the tool of causal diamonds. Causal diamonds describe the space-time region probed in an arbitrary experiment of given duration [17]. We find the beginning and end points of a brane observer's experiment that is just barely long enough to detect a given excitation in the bulk. The conjectured equivalence of bulk and boundary descriptions implies that the same experiment must also barely detect the dual CFT excitation on the brane. Hence the CFT state has support on the boundary of the associated causal diamond.

In section 6, we apply the causal diamond method to branes in AdS_5. In section 6.1, we consider the RS brane. We verify that the CFT image predicted by causal diamonds agrees with previous results obtained by more elaborate methods. The image is a shell, whose radius grows with the distance of a bulk excitation from the brane. In section 6.2, we find the same qualitative behavior for the CFT dual of excitations near an AdS_4 brane. However, we find that the CFT shell becomes infinitely large when the bulk excitation is still only a finite distance from the brane. This occurs precisely when the excitation reaches the boundary of the holographic domain calculated in section 4.

This provides a beautiful consistency check, and it answers the question raised earlier. When the particle exits the brane's holographic domain, the holographic image disappears from the AdS_4 brane by moving off to spatial infinity. In section 7, we consider the global structure of the AdS_4/AdS_5 system, which suggests an intriguing interpretation (section 8). Unlike the RS model, the AdS_4/AdS_5 system retains half of the boundary of AdS_5. Its holographic domain is evidently the remaining portion of the bulk that was not covered by the brane's holographic domain. As a particle crosses back and forth between domains, the holographic image must cross between the CFT on the brane, and the CFT on the AdS_5 half-boundary. We also discuss implications for the break-down of four-dimensional gravity on the brane.

2. Branes and space-like holography in AdS_5

In this section, we present metrics for flat and AdS_4 branes in AdS_5. In each case we relate brane volumes to bulk volumes. We examine the compatibility of this space-like relation with independent results for the strength of four-dimensional gravity, and with the holographic principle. In the RS case, we find no contradictions. In the AdS_4 case, we find a discrepancy of the space-like relation with the finiteness of the four-dimensional Planck mass, and we expose a violation of the holographic principle.

2.1 RS brane

Newton's constant, $G_\mathrm{N} = M_4^{-2}$, need not vanish even in the presence of non-compact extra dimensions. Consider five-dimensional gravity[3] with a negative cosmological constant, coupled to a three-brane of tension T:

$$S = \frac{M_5^3}{16\pi} \int d^5x \sqrt{-g} \left(R_5 + \frac{12}{\ell^2} \right) - T \int d^4x\, dr\, \delta(r-c) \sqrt{-\det g_{ij}}\,. \tag{2.1}$$

The Randall-Sundrum model [12] embeds a 3+1 dimensional Minkowski brane into portions of 4+1 dimensional anti-de Sitter space (AdS_5). This requires that the tension takes the value

$$T = \frac{3M_5^3}{4\pi\ell}\,. \tag{2.2}$$

The solution can be written as a warped product metric,

$$\frac{ds^2}{\ell^2} = e^{2r} \left(-dt^2 + d\rho^2 + \sin^2\rho\, d\Omega_2^2 \right) + dr^2\,, \tag{2.3}$$

where ℓ is the curvature scale of the AdS_5 space-time. The brane resides at an arbitrary fixed value of the warp coordinate, $r = c$. (Conventionally $r = 0$ is chosen, but with a view to the AdS_4 case we will find instructive to leave c unspecified.) The portion of the above metric with $r > c$ is removed, and one can think of the brane as a ridge at which the remaining portion of the AdS_5 Poincaré patch, $-\infty < r < c$, is matched to another copy of itself. We will consider the two copies to be identified under $\mathbb{Z}_2$.

Let us pick a finite three-dimensional spatial region V_3 at a fixed time t on the brane (figure 1). We associate to it a four-dimensional spatial volume V_4 by warped multiplication,

[3] We use the conventions of Wald [18] for the metric signature and the Ricci scalar. The quantity "λ" in [14] corresponds to $4\pi T/M_5^3$ in our notation.

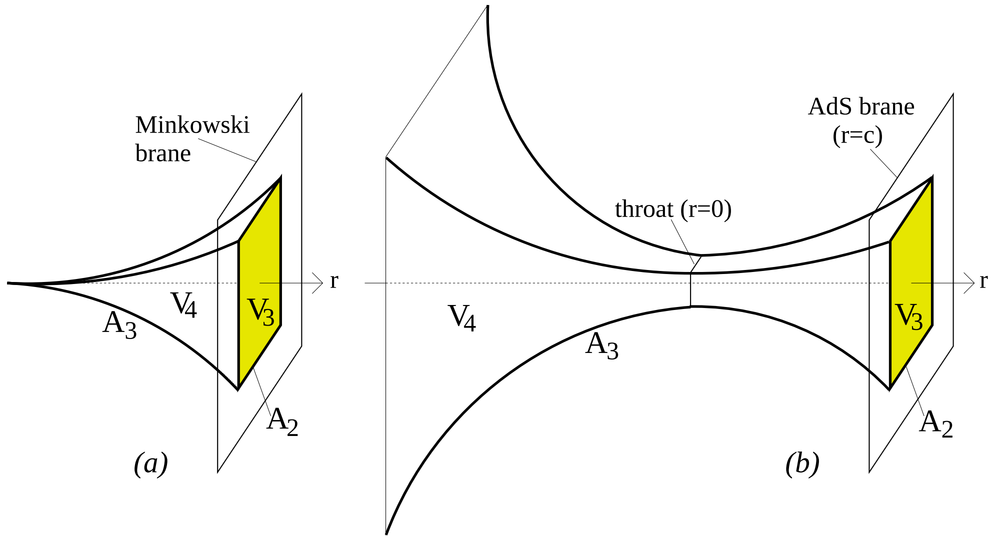

Figure 1: Branes at $r = c$ in a warped compactification of AdS_5; the time direction is suppressed. (a) A Minkowski brane has warp factor e^r. A finite volume V_3 on the brane corresponds to a finite volume V_4 in the bulk; similarly for areas A_2 and A_3. (b) An AdS_4 brane has warp factor $\cosh r$. Beyond the throat ($r = 0$) the bulk area and volume diverge. The physics in the infinite region V_4 cannot be holographically represented by CFT excitations in V_3.

i.e., by lifting the restriction $r = c$.[4] Even though the r coordinate extends an infinite proper distance away from the brane, this bulk volume will be finite:

$$V_4 = \int_{-\infty}^{c} \ell dr\, V_3 e^{r-c} = V_3 \ell\,. \tag{2.4}$$

In standard Kaluza-Klein compactification, the size of the extra dimension determines the four-dimensional Planck scale in terms of the five-dimensional one. The relation can be written in the form

$$\frac{M_4^2}{M_5^3} = \frac{V_4}{V_3}\,, \tag{2.5}$$

suggestive of the dilution of the gravitational field in the extra dimension. One might expect that (2.5) also applies to warped compactifications such as the RS model. Then (2.4) implies the relation

$$M_4^2 = M_5^3 \ell \tag{2.6}$$

between Planck masses in the RS model. This is indeed the relation found in [12] by analysis of graviton modes. We conclude that the association of bulk and brane regions by warped multiplication is consistent with the independently computed strength of four-dimensional gravity on the brane.

[4]Warped multiplication is clearly the most natural way to associate brane and bulk volumes at equal time. Nevertheless, one should keep in mind that it is based on a coordinate choice. Strictly speaking, the space-like relation between brane and bulk volumes is therefore not even uniquely defined. By contrast, the covariant relation via light-sheets is unambiguous (section 3).

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Space-like holography demands that CFT states with support on V_3 encode all bulk information in V_4, requiring in particular that

$$S_5(V_4) \leq \frac{V_3 M_5^3}{4} \,. \tag{2.7}$$

The entropy S_5 in the bulk region V_4 must not exceed the "area" V_3, in five-dimensional Planck units.

Since V_3 does not enclose V_4, we cannot use the space-like condition to argue that (2.7) follows from the covariant holographic principle. Indeed, after complementing the geometry in figure 1a with its mirror image across the brane, we recognize that V_3 is not even a portion of the boundary of V_4. The actual boundary of V_4, A_3, resides mostly in the bulk. Since A_3 is a closed surface around V_4 (we may neglect the point at infinity), it satisfies the space-like condition. It follows that

$$S_5(V_4) \leq \frac{A_3 M_5^3}{4} \,. \tag{2.8}$$

This inequality is stronger than (2.7), as we show next.

Let A_2 be the intersection of A_3 with the brane. Then A_3 is related to A_2 by warped multiplication:

$$A_3 = \int_{-\infty}^{c} \ell dr \, A_2 e^{r-c} = A_2 \ell \,. \tag{2.9}$$

Note that A_2 is the boundary of the brane region V_3. Moreover, we may assume that V_3 has no dimensions smaller than the CFT cut-off scale, ℓ; otherwise, CFT states cannot be localized to V_3. It follows that $V_3 > A_2\ell = A_3$. With (2.8) this implies the inequality (2.7). We conclude that space-like holography, as defined by projection along the warp direction r, is consistent with the holographic entropy bound in the RS model.

We note parenthetically that (2.9) implies an interesting statement, which however will not be needed later. Since the four-dimensional Planck mass is finite, the holographic principle must also apply to the four-dimensional theory on the brane:

$$S_4(V_3) \leq \frac{A_2 M_4^2}{4} \,. \tag{2.10}$$

Moreover, we assume that the CFT encodes all information in the bulk, i.e., $S_4(V_3) = S_5(V_4)$. Then (2.9) and (2.5) imply the equivalence of (2.8) and (2.10). The holographic entropy bound is equivalent in the four- and the five-dimensional theory, as it should be.

2.2 AdS branes and the break-down of space-like holography

The de-tuning of the brane tension (2.2) leads to generalizations of the RS model [19, 20, 21, 22, 14] in which the brane is no longer flat. Instead, the brane resides at constant r in a warped slicing of an AdS_5 bulk into dS_4 or AdS_4 slices. The latter case is of particular interest because the warp factor, $\cosh r$, fails to be monotonic:

$$\frac{ds^2}{\ell^2} = \cosh^2 r \left(-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_2^2\right) + dr^2 \,. \tag{2.11}$$

An AdS_4 brane of positive tension $T = \frac{3M_5^3}{4\pi\ell}\tanh c$ resides at $r = c > 0$. The portion of AdS_5 with $r > c$ is removed. We refer to the AdS_4 hypersurface $r = 0$ as the *throat* of the warped geometry, because the warp factor has a minimum there. Near the brane, between $r = c$ and the throat, the spatial bulk geometry shrinks away from the brane. This mimics the e^r behavior of the warp factor in the RS model.

Beyond the throat, from $r = 0$ to $r = -\infty$, the warp factor increases without bound (figure 1). Hence, the bulk volume associated with a given volume on the brane diverges:

$$V_4 = \int_{-\infty}^{c} \ell dr\, V_3 \frac{\cosh r}{\cosh c} \to \infty\,. \tag{2.12}$$

Reasoning analogous to (2.5) would suggest that the four-dimensional Planck mass should diverge. A mode analysis has shown, however, that gravity nevertheless localizes on the brane [14], with $M_4^2 \approx M_5^3\ell$. The covariant methods developed in section 5 resolve this contradiction, as we discuss at the end of section 7. We will find that gravity looks four-dimensional only for a limited time. The relevant time scale is the AdS_4 curvature radius

$$\ell_4 = \ell\cosh c\,, \tag{2.13}$$

which can be made parametrically larger than the AdS_5 radius, ℓ, by choosing $c \geq 1$.[5]

The space-like holographic principle, inequality (2.7), is clearly violated in the $\mathrm{AdS}_4/\mathrm{AdS}_5$ system. The "area" V_3 is finite, but the divergent bulk volume V_4 can contain arbitrarily large systems with entropy

$$S_5(V_4) \gg V_3 M_5^3\,. \tag{2.14}$$

Hence, the bulk physics cannot be equivalent to any field theory on the AdS_4 brane. This appears to conflict with the effective compactification to four dimensions indicated by the mode analysis of [14].

Next we review the covariant form of the holographic principle. In section 4, we will show how it resolves the paradox (2.14).

3. Light-sheet holography

The apparent contradictions we have found originate from our attempt to relate brane and bulk volumes in a naive, space-like manner. In this section we review the covariant entropy bound, a more precise and general formulation of the holographic principle. We describe how to calculate the extent of the holographic region bordering on a given area. We also discuss how to apply the covariant formalism to a brane of codimension 1.

3.1 Covariant entropy bound

The covariant formulation of the holographic principle [4, 5, 6] associates bulk regions to areas. In D space-time dimensions, by an area we mean a surface of $D - 2$ spatial dimensions, i.e., *not* the history of a $D - 2$ surface.

[5] This corresponds to an appropriate choice of brane tension, which we treat as an adjustable parameter of the theory. We will not be concerned with the microscopic origin of these scales because that problem seems unrelated to the questions addressed here. See [23, 15] for a recent approach.

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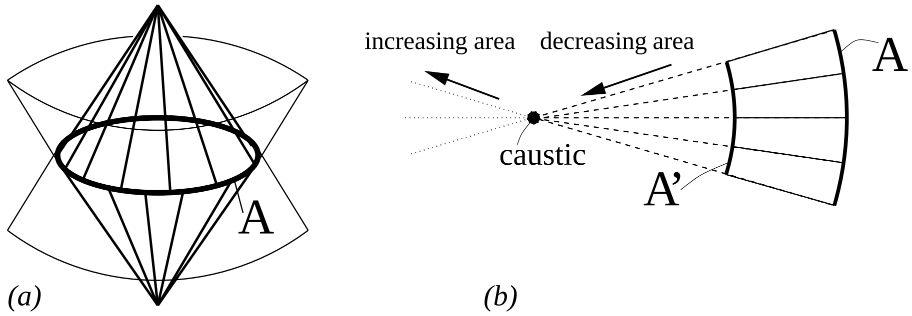

Figure 2: (a) A space-time picture of the two light-sheets of a spherical area A. The entropy on either cone is bounded by A. The other two light-like directions are the skirts drawn in thin outline. Their cross-sectional area is increasing, so the entropy of the skirts is not related to A. (b) (Here the time direction is suppressed.) The requirement of decreasing area, $A' < A$, is a local condition. When neighboring light-rays intersect, the light-sheet must be terminated.

There are four families of light-rays orthogonal to any area A, as shown in figure 2: two to the past, and two to the future. Consider one such family, with affine parameter λ along the light-rays. The cross-sectional area $A'(\lambda)$ spanned by the light-rays will change as one follows the rays away from A. By continuity, $A'(\lambda)$ must be non-increasing for at least two of the four families. The hypersurfaces generated by the light-rays pointing in any one of these "shrinking" directions are called *light-sheets.*

Thus, light-sheets are null hypersurfaces of dimension $D-1$. They would represent an equal-time hypersurface if we used light-cone coordinates. In fact, they represent a way of taking a snapshot of matter systems. The holographic principle states that the number of degrees of freedom, and hence the entropy, on any light-sheet of A is bounded by $A/4$ in Planck units:

$$S(\text{lightsheet of } A) \leq \frac{A M_D^{D-2}}{4}\,, \tag{3.1}$$

where M_D is the Planck mass in the D-dimensional theory.

Formally, light-sheets are characterized by

$$\theta \leq 0\,. \tag{3.2}$$

The expansion, θ, of a family of light-rays, defined in (3.8) below, is a local measure of the rate of change of the infinitesimal area element spanned by neighboring light-rays. The condition (3.2) generalizes the usual notion of "inside". It is well-defined where conventional notions fail, as in closed or highly dynamical geometries, and for open surfaces. For convex closed surfaces in weakly gravitating regions of asymptotically flat space, it reproduces the intuitive answer: inside is where infinity is not. The two families that sweep through the enclosed ("inside") region, one past-directed and one future-directed, are indeed contracting. They form a past and a future light-cone ending on A. The light-rays going to the outside have growing cross-sectional area and must be excluded from consideration (figure 2).

The local condition (3.2) also determines where light-sheets must be terminated. The light-rays may be followed only as long as the expansion remains non-positive [4]. If the null energy condition,

$$T_{ab}k^a k^b \geq 0 \qquad \text{for all null vectors } k^a\,, \tag{3.3}$$

is satisfied, the expansion can only become positive after diverging to $-\infty$. This happens at caustics, when neighboring light-rays intersections Hence, light-sheets must be terminated at caustics, such as the tips of the light-cones in figure 2.

3.2 Finding caustics

In this subsection we show how to calculate the location of caustics on a light-sheet of a given area A.

Let A be a surface of $D-2$ spatial dimensions, parametrized by coordinates x^α, $\alpha = 1, \ldots, D-2$. Pick one of the four families of light-rays that emanate from A into the past and future directions to either side of A. Each light-ray satisfies the equation for geodesics:

$$\frac{dk^a}{d\lambda} + \Gamma^a{}_{bc}k^b k^c = 0\,. \tag{3.4}$$

where λ is an affine parameter. The tangent vector k^a is defined by

$$k^a = \frac{dx^a}{d\lambda} \tag{3.5}$$

and satisfies the null condition $k^a k_a = 0$.

Let l^a be the null vector field on A that is orthogonal to A and satisfies $k^a l_a = -2$. (This means that l^a has the same time direction as k^a and is tangent to the orthogonal light-rays constructed on the other side of A.) The induced $D-2$ dimensional metric on the surface A is given by

$$h_{ab} = g_{ab} + \frac{1}{2}\left(k_a l_b + k_b l_a\right). \tag{3.6}$$

In a similar manner, an induced metric can be found for all other cross-sections of the light-sheet. The *null extrinsic curvature*

$$B_{ab} = h^c{}_a h^d{}_b \nabla_c k_d \tag{3.7}$$

contains information about the expansion, θ, shear, σ_{ab}, and twist, ω_{ab}, of the light-sheet:

$$\theta = h^{ab}B_{ab}\,, \tag{3.8}$$

$$\sigma_{ab} = \frac{1}{2}\left(B_{ab} + B_{ba}\right) - \frac{1}{D-2}\theta h_{ab}\,, \tag{3.9}$$

$$\omega_{ab} = \frac{1}{2}\left(B_{ab} - B_{ba}\right). \tag{3.10}$$

The Raychaudhuri equation describes the change of the expansion along the light-rays:

$$\frac{d\theta}{d\lambda} = -\frac{1}{D-2}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - 8\pi G_{\mathrm{N}} T_{ab}k^a k^b\,. \tag{3.11}$$

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For a surface-orthogonal family of light-rays, the twist vanishes. The final term, $-T_{ab}k^a k^b$, will be non-positive if the null energy condition is satisfied by matter. Then the right hand side of the Raychaudhuri equation is manifestly non-positive, and one can show the following lemma: if the expansion at $\lambda = \lambda_0$ takes on the non-zero value θ_0, then there is a caustic (i.e., $|\theta| \to \infty$) somewhere between λ_0 and $\lambda_0 - \frac{D-2}{\theta_0}$. This statement holds separately for each light-ray in the family; in general, $\theta_0 = \theta_0(x^\alpha)$.

Note that we can take (λ, x^α) to parameterize simultaneously two of the four families of light-rays orthogonal to A; one family has $\lambda > \lambda_0$ while the other has $\lambda < \lambda_0$. We have defined θ to describe the expansion of the former family. Since $\theta_0 \to -\theta_0$ under $\lambda \to \lambda_0 - \lambda$, we now see explicitly that at least one of the pair forms a light-sheet. For $\theta_0 \neq 0$, the lemma shows that there will necessarily be a caustic on the light-sheet.

To obtain the exact caustic position for a given matter distribution T_{ab}, one would have to solve the Raychaudhuri equation and a coupled evolution equation for the shear:

$$k^c \nabla_c \sigma_{ab} = -\theta \sigma_{ab} + h^e{}_a h^f{}_d C_{cbef} k^c k^d \,. \tag{3.12}$$

The calculation simplifies under certain conditions, which will be seen to hold in the example studied in section 4. Suppose that the Weyl tensor, C_{cbef}, vanishes everywhere on the light-sheet. Suppose further that the shear vanishes initially: $\sigma_{ab}(\lambda_0) = 0$. Then it follows from (3.12) that the shear vanishes everywhere on the lightsheet. If the further condition

$$\theta^2(\lambda_0) \gg G_{\mathrm{N}} T_{ab} k^a k^b \tag{3.13}$$

holds for all λ, the effect of matter on the focusing is negligible compared to the non-linear term, and (3.11) becomes

$$\frac{d\theta}{d\lambda} = -\frac{1}{D-2}\theta^2 \,. \tag{3.14}$$

It follows that the caustics' location depends only on the initial expansion and is given by

$$\lambda_b(x^\alpha) = \lambda_0 - \frac{D-2}{\theta_0(x^\alpha)} \,. \tag{3.15}$$

Generically the caustics form a surface of the same dimension as A.

Once the affine parameter at each caustic is known, its space-time coordinates can be found by solving the null geodesic equation (3.4).

3.3 Holographic domains

In the RS model and the generalizations we consider, the brane occupies a time-like hypersurface H_{D-1} of AdS_5. But to construct a light-sheet, we need to start from a spatial surface of codimension 2. Such surfaces can be obtained by slicing the brane into surfaces of equal time, $V_{D-2}(t)$, as shown in figure 3. For each $V_{D-2}(t)$ we can construct a light-sheet $L_{D-1}(t)$. The resulting sequence of $(D-1)$ dimensional light-sheets will foliate a D-dimensional portion of the bulk, the *holographic domain* $\mathcal{D}(H_{D-1})$ associated with the brane H_{D-1}.

In general, the location of caustics will depend on the choice of slicing, which we have not yet specified. A different foliation $V'_{D-2}(t')$ will have different light-sheets $L'_{D-2}(t')$,

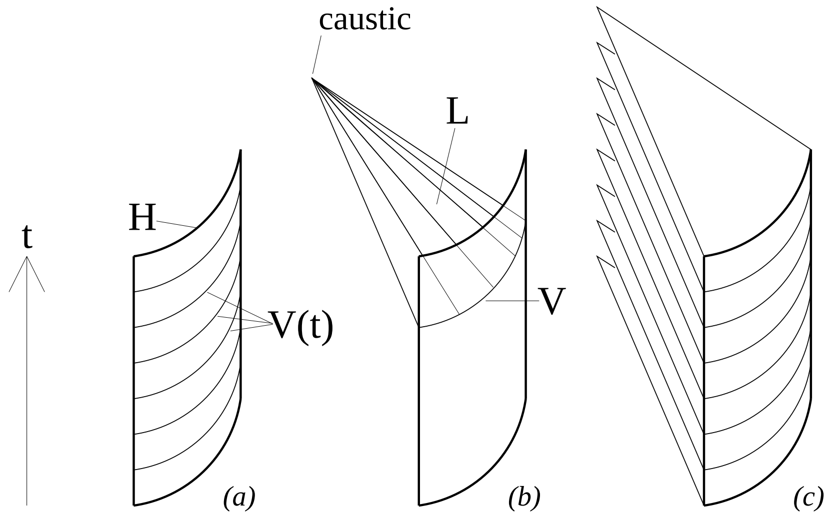

Figure 3: Constructing the holographic domain $\mathcal{D}$ of a brane of codimension 1, H_{D-1}. (a) Slice H_{D-1} into equal time surfaces, $V_{D-2}(t)$ (b) Each $V_{D-2}(t)$ is an area from the bulk point of view. Construct its light-sheet, $L_{D-1}(t)$. (c) The light-sheets fill out a portion of the bulk, $\mathcal{D}(H_{D-1})$.

and would yield a different answer for $\mathcal{D}(H_{D-1})$. We also have not specified which of the two allowed light-sheets should be considered.

Here we propose to resolve both ambiguities by demanding time reversal invariance. In the cases of interest the brane slices are *normal* [4], that is, they possess a past and a future light-sheet going to the same side. For a given slicing $V_{D-2}(t)$ let $\mathcal{D}^+[V_{D-2}(t)]$ ($\mathcal{D}^-[V_{D-2}(t)]$) be the region foliated by the future (past) light-sheets $L^+_{D-1}(t)$ ($L^-_{D-1}(t)$). One might declare, say, that the holographic domain is always defined by past light-sheets ($\mathcal{D} = \mathcal{D}^-$). But then the brane would encode a different bulk region after time reversal, even though the CFT is time-reversal invariant.

This motivates the demand that the slicing $V_{D-2}(t)$ be chosen so that

$$\mathcal{D}^+[V_{D-2}(t)] = \mathcal{D}^-[V_{D-2}(t)] \equiv \mathcal{D}(H_{D-1})\,. \tag{3.16}$$

In other words, we require that it should not matter whether past or future light-sheets are used to construct the holographic domain.[6] We will not analyze here whether this condition is appropriate for arbitrary brane and bulk metrics. It is a reasonable condition for the present purpose.

For AdS_4 and RS branes in AdS_5, we can find suitable slices V_{D-2} by demanding that the null extrinsic curvature of the future-directed light-rays, B^+_{ab}, should be the same as that of the past-directed light-rays, B^-_{ab}, at every point x^α on V_{D-2}:

$$B^+_{ab}(x^\alpha) = B^-_{ab}(x^\alpha)\,. \tag{3.17}$$

[6] $\mathcal{D}^+$ and $\mathcal{D}^-$ may contain the same information even if they differ. For example, suppose that $\mathcal{D}^+ = \mathcal{D}^- - Q$, where the points in Q are causally independent of any points outside $\mathcal{D}^+$. This motivates the refined condition $D^\pm\{\mathcal{D}^+[V_{D-2}(t)]\} = D^\pm\{\mathcal{D}^-[V_{D-2}(t)]\}$, where $D^\pm(U)$ denotes the union of the past and future domains of dependence of the set U [24]. This refinement will not play a role in the examples we study.

This is equivalent to the condition that the slices $V_{D-2}(t)$ have vanishing (ordinary) extrinsic curvature in H:

$$h_a^{\ c}\, \nabla_c t_b = 0\,. \tag{3.18}$$

Here t^c is the future-directed unit normal vector of $V_{D-2}(t)$ in H_{D-1}.

For $H = \mathrm{AdS}_4$, (3.18) is uniquely satisfied by the hyperbolic space

$$\frac{ds_3^2}{\ell_4^2} = d\rho^2 + \sinh^2\rho\, d\Omega_2^2\,. \tag{3.19}$$

Any complete foliation of H by such spaces will satisfy the condition (3.16) and will give the same answer for $\mathcal{D}$. The simplest example is given by the "global coordinates" of AdS_4,

$$\frac{ds_4^2}{\ell_4^2} = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_2^2\,. \tag{3.20}$$

We have chosen the AdS_5 metric (2.11) to conform to these coordinates.

For $H = \mathrm{M}^4$, (3.18) is satisfied by the flat $\mathbb{R}^3$ slices arising in the metric (2.3).

4. Holographic domains in AdS_5

In this section we locate the caustics of light-sheets of a given brane in AdS_5. This determines the extent of the brane's holographic domain in the bulk. We describe the calculation in detail for an AdS_4 brane and also state the result for the RS brane.

The metric of an AdS_4 brane immersed in AdS_5 was given in (2.11):

$$\frac{ds^2}{\ell^2} = \cosh^2 r\left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_2^2\right) + dr^2\,. \tag{4.1}$$

The time slicing induced on the brane, at $r = c$, obeys the condition (3.18). Because the metric is static, it will suffice to consider the future-directed light-sheet of one time-slice, $t = 0$. We will compute the caustic location, $r = b$. The holographic domain associated to the brane will be characterized by $b \leq r \leq c$.

Following the general method described in section 3, we will first determine the caustic position in terms of the affine parameter λ; then we will find $r(\lambda)$.

The null vector fields k^a and l^a, defined by

$$k^t = \frac{1}{\cosh r \cosh\rho}\,, \quad k^r = -1\,, \quad k^\rho = k^\theta = k^\phi = 0 \tag{4.2}$$

and

$$l^t = \frac{1}{\cosh r \cosh\rho}\,, \quad l^r = 1\,, \quad l^\rho = l^\theta = l^\phi = 0 \tag{4.3}$$

satisfy $k^a l_a = -2$ and are orthogonal to the brane slice defined by $r = c$, $t = 0$. At this slice, the vector field k^a (l^a) coincides with the tangent vector field of the future-directed orthogonal light-rays going in the negative (positive) r direction. Keeping in mind that the region $r > c$ is removed from (4.1) and replaced by a second copy of the geometry, we need only consider the light-rays with tangent vector k^a, which we will find to have negative expansion. Note that (4.2) and (4.3) are not necessarily orthogonal to any other cross-sections of the light-sheet, but they will allow the computation of the initial expansion and shear.

According to (3.6), the induced metric on the brane slice is given by

$$h^a{}_b = \mathrm{diag}(0,0,1,1,1)\,. \tag{4.4}$$

With (3.7) one finds

$$B^a{}_b = -\tanh c\;\mathrm{diag}(0,0,1,1,1)\,. \tag{4.5}$$

Hence, the initial expansion of the light-sheet is independent of the position along the brane slice:

$$\theta_0 = -3\tanh c\,. \tag{4.6}$$

As required, the twist vanishes. We note that the initial shear also vanishes. Moreover, the Weyl tensor C_{cbad} vanishes in vacuum AdS_5. We conclude that the shear vanishes everywhere on the light-sheet. For $c > 1$, the initial expansion is of order unity, so that condition (3.13) is guaranteed to hold in the weak field limit. Hence, the location of the caustic is determined by the initial expansion alone. The value of the affine parameter at the brane may be set to zero. Then (3.15) yields

$$\lambda_b = \coth c\,. \tag{4.7}$$

The geodesic equation (3.4) for the r coordinate is

$$\frac{d^2r}{d\lambda^2} + \sinh r\cosh r\left[\cosh^2\rho\left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{d\rho}{d\lambda}\right)^2\right] = 0\,. \tag{4.8}$$

The tangent vector is null:

$$\cosh^2\rho\left(\frac{dt}{d\lambda}\right)^2 - \frac{1}{\cosh^2 r}\left(\frac{dr}{d\lambda}\right)^2 - \left(\frac{d\rho}{d\lambda}\right)^2 = 0\,. \tag{4.9}$$

Hence, the r equation simplifies to

$$\frac{d^2r}{d\lambda^2} + \tanh r\left(\frac{dr}{d\lambda}\right)^2 = 0\,, \tag{4.10}$$

with general solution

$$\lambda(r) = C_1\sinh r + C_2\,. \tag{4.11}$$

At $r = c$, one has $\lambda = 0$ and $k^r = \frac{dr}{d\lambda} = -1$, which fixes the constants:

$$\lambda(r) = \frac{\sinh c - \sinh r}{\cosh c}\,. \tag{4.12}$$

The caustic lies at $r = b$ with $\lambda(b) = \lambda_b$. Hence,

$$\sinh b = \frac{-1}{\sinh c}\,. \tag{4.13}$$

The calculation involved the assumption $c \geq 1$, which is also required for the localization of gravity over a significant scale [14].

Notice that we have shown that each light-ray in the light-sheet terminates at $r = b \approx -2e^{-c}$, just beyond the throat of the warp factor. We conclude that the holographic domain $\mathcal{D}(H)$ of an AdS_4 brane H at $r = c > 1$ is the bulk region

$$b < r < c \tag{4.14}$$

in the coordinates (2.11).[7] The infinite region $r < b$ is not included on the light-sheet. This eliminates the possibility of violating the holographic principle with large entropic systems in the region behind the throat.

An analogous but simpler calculation of light-sheets of the RS brane shows that their caustics lie on the Poincaré horizon. Hence, the holographic domain of the RS brane is the surviving portion of the Poincaré patch, i.e., the portion of AdS_5 satisfying

$$r < c \tag{4.15}$$

in the coordinates (2.3).

5. Holographic images from causal diamonds

Bulk physics taking place within a brane's holographic domain $\mathcal{D}$ should have a holographic dual in terms of CFT excitations on the brane; bulk physics outside $\mathcal{D}$ should not. This raises a sharp question, which we study in the next two sections. How does the holographic image disappear from the brane when a bulk particle exits from $\mathcal{D}$?

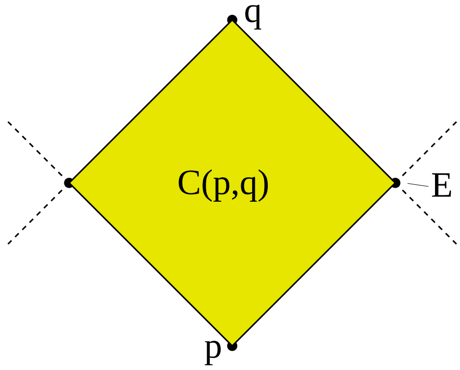

Figure 4: A causal diamond $C(p, q)$ is the space-time region probed by an experiment beginning at p and ending at q.

In this section, we give a simple construction that allows us to locate on the brane the CFT excitations dual to a bulk particle. We will apply this construction to the RS and KR models in the following section. Our construction is general and, in certain limits, can also be applied in the AdS/CFT correspondence (unmodified by branes).[8]

Consider a pointlike observer in an arbitrary space-time. Suppose the observer performs an experiment that begins at a point p on his world-line and ends at some later point q (figure 4). By causality, this experiment cannot probe points that lie outside the past light-cone of q, because light does not have enough time to reach the observer. Moreover, no points outside the future light-cone of p can be probed. Hence, the experiment

[7] We have shown only that $\mathcal{D}(H)$ is contained in the region (4.14). By considering a past light-cone from $r = b$, $\rho = 0$, whose intersection with the hypersurface $r = b$ meets H at their common spatial infinity, one can show that $\mathcal{D}(H)$ is in fact equal to the region $b < r < c$.

[8] Causal diamonds have previously arisen in connection with the AdS/CFT correspondence in refs. [25, 26, 27]. We would like to thank S. Ross for bringing those references to our attention. An important early investigation of causality in the AdS/CFT correspondence [28] was restricted to a set of questions that did not require the use of causal diamonds.

can be associated to the *causal diamond* [29], $C(p,q)$, defined as the space-time region that lies both in the future of p and in the past of q. It is bounded by light-cones of p and q, whose intersection we call the *edge*, $E(p,q)$.

Any experiment beginning at p and ending at q can only learn directly about physics taking place in the space-time region $C(p,q)$. Indirectly, of course, inferences can be made about the previous history of particles that entered the causal diamond from the past. But for the purpose of estimating the amount of information that can be gained in such an experiment, it suffices to know the maximum entropy that can be contained within $C(p,q)$.

An observer trapped on a brane in AdS_5 has two ways of describing certain experiments. A probe of the state of the holographic CFT on the brane can alternatively be described as an excursion into the bulk. These languages must be equivalent if the holographic duality is faithful. By causality, an experiment of finite duration cannot probe bulk regions arbitrarily far from the brane. It follows that the CFT will not contain information about distant bulk regions when probed at a sufficiently short distance scale.

More specifically, given a five-dimensional causal diamond $C_5(p,q)$ with p and q on the brane, one can associate to it a four-dimensional causal diamond $C_4(p,q)$, obtained by intersecting $C_5(p,q)$ with the brane (figure 5). Let $V_4(p,q)$ be a bulk volume bounded by the edge of $C_5(p,q)$. Let $V_3(p,q)$ be the intersection of $V_4(p,q)$ with the brane. Hence, $V_3(p,q)$ is a brane volume bounded by $E_2(p,q)$, the edge of $C_4(p,q)$. By causality, excitations of the CFT with support on $V_3(p,q)$ should encode no more nor less than the bulk physics taking place in the region $V_4(p,q)$.

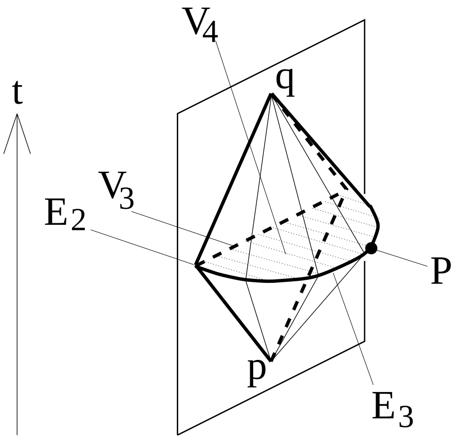

Figure 5: Causal diamond barely containing a bulk event at P. By causality, the holographic image of P has support on the intersection of the edge with the brane. Hence, it is shell of CFT energy whose radius grows with the distance of P from the brane.

We are interested specifically in the support $\mathcal{S}(P)$ of the CFT state dual to a pointlike excitation in the bulk. This can be constructed as follows. Given an excitation at a point P off the brane, find points p and q on the brane such that $C_5(p,q)$ contains P but does not contain points farther from the brane than P. By causality, the CFT dual to the event at P must have support only on $V_3(p,q)$:

$$\mathcal{S}(P) \subset V_3(p,q)\,. \tag{5.1}$$

Now consider $C_5(p',q')$, where p' (q') are brane events infinitesimally later (earlier) than p (q). This causal diamond is nested just within $C_5(p,q)$ and barely fails to include P. In other words, the (p',q') experiment fails to detect P in the bulk. Then causality demands that it must also fail to probe the holographic image of P on the brane. The (p',q') experiment probes the brane volume $V_3(p',q')$. It follows that the CFT dual to P has no support on $V_3(p',q')$:

$$\mathcal{S}(P) \cap V_3(p',q') = \emptyset\,. \tag{5.2}$$

Combining (5.1) and (5.2), we conclude that

$$\mathcal{S}(P) \subset V_3(p,q) - V_3(p',q') = E_2(p,q)\,. \tag{5.3}$$

Physically, the edge $E_2(p,q)$ is the distance scale explored on the brane in an experiment that probes as far as to the point P into the bulk. The CFT excitation dual to an event P in the bulk (the *holographic image* of P) has support on the edge of the causal diamond $C_4(p,q)$.

6. Evanescence of CFT shells

In this section we check the consistency of our calculation of the holographic domain, Eqs. (4.14) and (4.15). Suppose a local bulk excitation moves away from the brane and eventually exits the holographic domain of the brane. We would like to verify that the holographic image disappears from the brane at this moment. For this purpose we apply the method of causal diamonds to locate holographic images for RS and AdS_4 branes in AdS_5. We will show that the image moves off to infinity as the corresponding bulk source exits the brane's holographic domain.

Recall from the metrics (2.3) and (2.11) that ρ is a radial coordinate on the brane in both cases. The coordinate r parametrizes the warped direction away from the brane. The brane resides at $r = c$. We consider a bulk event P at $r = r_P < c$. The translation symmetries in the brane directions allow us to take P to be at $\rho = 0$, $t = 0$.

We must find a causal diamond $C_5(p,q)$ that contains P and contains no points farther from the brane than P. This does not completely fix p and q. It implies only that they will be at the same value of ρ and at antipodal angles on the sphere. The boost symmetry of the brane, however, allows us to take p and q to reside at $\rho = 0$. We will determine their time coordinates, $\mp\Delta t(r_P)$, by computing the time coordinate when a past (future) light-ray from P along $\rho = 0$ reaches the brane at $r = c$.

The holographic image of P lies on the edge of the causal diamond $C_4(p,q)$. The edge, $E_2(p,q)$, consists of the outermost points on the brane reached by light-rays that start at p and are reflected back to q. Hence, $E_2(p,q)$ is a shell of radius $\rho(r_P)$, where $\rho(r_P)$ is the coordinate distance traveled by a light-ray on the brane in the coordinate time $\Delta t(r_P)$.

6.1 RS brane

In the RS model, described by the metric (2.3), a bulk light-ray between P and q obeys

$$\int_0^{\Delta t(r_P)} dt = \int_{r_P}^{c} e^{-r} dr\,. \tag{6.1}$$

An analogous equation governs the light-ray from p to P. Solving for $\Delta t(r_P)$ one finds

$$\Delta t(r_P) = e^{-r_P} - e^{-c}\,. \tag{6.2}$$

Next we consider the light-rays on the brane that radiate out from p and reflect on the edge $E_2(p,q)$ so as to focus at q. The rays returning from $E_2(p,q)$ to q obey

$$\int_0^{\rho(r_P)} dr = \int_0^{\Delta t(r_P)} dt\,, \tag{6.3}$$

with a similar equation holding for the outgoing rays. It follows that

$$\rho(r_P) = \Delta t(r_P)\,. \tag{6.4}$$

By inserting (6.2) into (6.4) we obtain the desired relation between the position of a bulk event, r_P and the radius of its holographic image, $\rho(r_P)$:

$$\rho(r_P) = e^{-r_P} - e^{-c} \qquad \text{(RS brane)}\,. \tag{6.5}$$

This result, obtained only from considerations of causality, is consistent with more elaborate studies of freely falling particles in AdS_5 [30, 31]. A particle at r_P in the bulk corresponds holographically to CFT energy momentum localized on a shell of radius $\rho(r_P)$. As the particle moves deeper into the bulk, the shell expands. (We are working in the geometrical optics approximation. One would expect the width of the shell to be smeared by a characteristic wavelength no shorter than the ultraviolet cutoff of the CFT. Indeed, for a particle at rest when released from the brane, the width was found to be of order the AdS_5 radius, ℓ, in [30].) By repeating the above procedure for every point on a time-like bulk geodesic one finds that the shell expands with constant acceleration. Hence its world tube begins and ends on the null infinities of Minkowski space, $\mathcal{I}^-$ and $\mathcal{I}^+$.

This sheds light on an interesting aspect of the AdS/CFT correspondence. Recall that the holographic domain of a Minkowski brane, (4.15), is (the surviving portion of) the Poincaré region of AdS_5. In the full AdS/CFT correspondence, a similar relation holds between the (whole) Poincaré region and its conformal boundary. One might be concerned about the incompleteness of the Poincaré region. In the bulk, particles can enter through the past Poincaré horizon and disappear behind the future Poincaré horizon. Of course this is just a consequence of using a non-global coordinate system and does not imply loss of unitarity. But the Poincaré region is holographically dual to a field theory on a *complete* space-time, four-dimensional Minkowski space. How can the dual theory handle the loss and gain of excitations and yet be unitary?

As a particle approaches the Poincaré horizon ($r_P \to -\infty$) the shell on the brane becomes infinitely large at a rate approaching the speed of light. By energy conservation, it will also become infinitely dilute. This process takes an infinite proper time for a brane observer. At the classical level, we thus understand how the holographic image disappears when a particle exits the holographic domain of the RS brane. The boundary of the Poincaré region is mapped to the conformal boundary of Minkowski space. The loss and gain of states in the bulk corresponds to boundary conditions imposed at the null infinities, $\mathcal{I}^+$ and $\mathcal{I}^-$, of the RS brane.

6.2 AdS_4 branes

The causal diamond method is readily applied to the more complicated metric of the AdS_4/AdS_5 system. In this case one finds the relation

$$\arctan e^c - \arctan e^{r_P} = \frac{\Delta t(r_P)}{2} = \arctan e^{\rho(r_P)} - \frac{\pi}{4}\,. \tag{6.6}$$

This implies

$$\rho(r_P) = \ln \frac{1 + e^c - e^{r_P} + e^{c+r_P}}{1 - e^c + e^{r_P} + e^{c+r_P}} \,. \tag{6.7}$$

A useful equivalent expression is

$$\rho(r_P) = \ln \frac{\cosh x + \sinh y}{\cosh x - \sinh y} \,, \tag{6.8}$$

where

$$x = \frac{c + r_P}{2} \,, \qquad y = \frac{c - r_P}{2} \,. \tag{6.9}$$

It is easy to check that $\rho(r_P) = 0$ for $r_P = c$, and

$$\frac{d\rho(r_P)}{d(c - r_P)} > 0 \tag{6.10}$$

for $r_P < c$. Thus we find again that the image of a bulk event at P is a CFT shell whose size increases monotonically with the distance of the particle from the brane.

However, the functional relations (6.5) and (6.7) differ crucially in that the shell on the AdS_4 brane becomes infinitely large already for finite r_P. The function (6.7) diverges at $r_P = b_P$, where

$$\sinh b_P = \frac{-1}{\sinh c} \,. \tag{6.11}$$

Comparison with (4.13) reveals that $b_P = b$.

In other words, the CFT shell reaches the boundary of AdS_4 precisely when the bulk excitation reaches the caustic surface that delimits the holographic bulk domain associated to the AdS_4 brane! We have thus completed an important consistency check. We have related the escape from the holographic domain to the evanescence of the holographic image. The holographic image of a bulk particle exiting the brane's holographic domain, $\mathcal{D}(H)$, is a CFT shell moving off the brane.

7. Global holography of the $\mathrm{AdS}_4/\mathrm{AdS}_5$ system

After a particle leaves $\mathcal{D}(H)$, it is still in an AdS_5 bulk, and one would expect a kind of holographic duality to a CFT to persist. Since this CFT cannot be on the brane, it must reside elsewhere. In this section we find that simple bulk dynamics leads to an interesting interplay between two field theories, or equivalently, between a single field theory and its boundary.

For definiteness, let us consider a particle released at time $t = 0$ from the origin, $\rho = 0$, of an AdS_4 brane at $r = c$. The particle will fall freely in the bulk, which means that it will oscillate between $r = c$ and $r = -c$. Its frequency is ℓ^{-1} as measured at $r = 0$, or ℓ_4^{-1} as measured on the brane at $r = c$. In particular, coming from the brane, it will not stop and turn around when it reaches the boundary of $\mathcal{D}(H)$, at $r = b$. (In this section we will take $c \gg 1$ so that (4.13) can be approximated as $b \approx -e^{-2c} \approx 0$.) Hence, the holographic image on the AdS_4 brane, after moving off to spatial infinity, will *not* be reflected back into the brane.

This is significant, because it is common to impose reflecting boundary conditions in anti-de Sitter space-times. In the case of the AdS_4 brane, this would mean that the shell should be instantly reflected at spatial infinity and start to shrink back. We have just shown that this behavior would be inconsistent with the bulk dynamics. We will now show that consistent boundary conditions on the AdS_4 brane are obtained by coupling the brane to the remaining boundary of the AdS_5 space.[9]

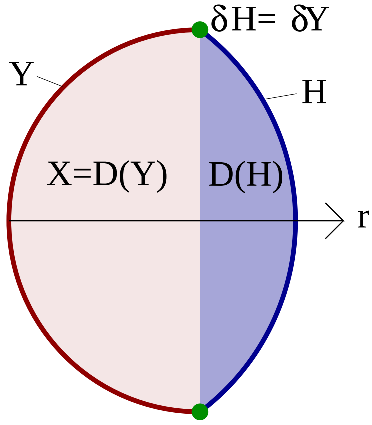

Figure 6: Constant time slice through the Penrose diagram of the $\mathrm{AdS}_4/\mathrm{AdS}_5$ system. The S^2 angular directions are suppressed for clarity. They can be partially recovered by rotating the diagram around the r axis. The diagram shows the brane, H, its holographic domain, $\mathcal{D}(H)$, the surviving portion of the AdS_5 boundary, Y, and its holographic domain, $\mathcal{D}(Y)$.

The metric (2.11) covers AdS_5 globally, with r ranging from $-\infty$ to ∞. In the $\mathrm{AdS}_4/\mathrm{AdS}_5$ system a brane resides at $r = c$. The bulk metric is still described by (2.11), but r ranges only from $-\infty$ to c (figure 6). In section 4, we showed that the portion $0 < r < c$ constitutes the holographic domain of the AdS_4 brane, $\mathcal{D}(H)$. The remaining region, X, is defined by $-\infty < r < 0$. X is half of AdS_5, and has a boundary, Y, at $r \to -\infty$. Y has topology $D^3 \times \mathbb{R}$, i.e., its spatial sections are three-dimensional conformal disks. Note that Y and H are joined on a surface $\partial Y = \partial H$ of topology $S^2 \times R$, defined by taking $\rho \to \infty$ at arbitrary fixed r.

It is convenient to regard Y as the $c' \to -\infty$ limit of an AdS_4 hypersurface at $r = c'$. By substituting c' for c in (4.13) and taking the limit, we find that the light-sheets of Y have caustics at $r = 0$. Hence, the holographic domain of Y is precisely the region X:

$$\mathcal{D}(Y) = X\,. \tag{7.1}$$

One would expect that an AdS/CFT duality relates bulk excitations with support in X to CFT states on Y.

We have used the light-sheet construction to show that the entire AdS_5 bulk is the disjoint union of the holographic domain of the brane, $\mathcal{D}(H)$, with the holographic domain of the remaining portion of the boundary, $\mathcal{D}(Y)$. Next, we will use causal diamonds to verify that a particle crossing the boundary between $\mathcal{D}(H)$ and $\mathcal{D}(Y)$, $r = 0$, corresponds to a CFT excitation crossing the boundary between H and Y, $\partial H = \partial Y$.

For a particle in $\mathcal{D}(H)$, we found in the previous section that its holographic image becomes larger on H and reaches ∂H, as the particle moves away from the brane and reaches $r = 0$. What remains to be verified is that the image of a particle in $\mathcal{D}(Y)$ lies on Y and approaches ∂Y as the particle approaches $r = 0$ from the other side.

[9]It was already observed in [14] that the $\mathrm{AdS}_4/\mathrm{AdS}_5$ system has a dual description in terms of two conformal field theories interacting on their intersection. This was also true of the intersecting brane configurations discussed in [15, 32]. What is new here is our use of causal diamonds, which allow us to describe details of the CFT dynamics dual to certain bulk processes.

Not only the light-sheet construction, but also the discussion of causal diamonds carries over to hypersurfaces at $r = c'$, with $c' \to -\infty$. Substituting in (6.6) we find that a bulk particle at $r = r_P < 0$ will be dual to a shell of radius

$$\rho(r_P) = \ln \coth \frac{|r_P|}{2} \tag{7.2}$$

on Y. When the particle approaches the boundary of $\mathcal{D}(Y)$, at $r = 0$, we find $\rho \to \infty$. Then the shell will be on ∂Y. This is significant, because the boundary of Y is also the boundary of the AdS_4 brane:

$$\partial Y = \partial H \,. \tag{7.3}$$

This allows the holographic image to cross over smoothly between Y and H as a bulk particle crosses $r = 0$.

Now let us return to the bulk particle oscillating between $r = \pm c$. Coming from the AdS_4 brane H, at $r = c$, it will fall towards $r = 0$. The holographic dual to this part of its path is a CFT shell expanding on H. The shell reaches infinity, ∂H, when the particle gets to the boundary of the brane's holographic domain, $\mathcal{D}(H)$, near $r = 0$. Then the particle crosses over to the bulk region $X = \mathcal{D}(Y)$, and the shell moves from ∂Y onto Y proper. The particle turns around at $r = -c$, when the shell is at $\rho = \ln \coth(c/2)$ on Y.

We now understand why reflecting boundary conditions are inappropriate for the spatial infinity of the AdS_4 brane. The brane H is coupled at ∂H to a portion of the conformal infinity of AdS_5, Y, which can be thought of as the limit of another AdS_4 space. Instead of reflecting at ∂H, CFT data first complete a half-period in Y before re-entering H. It will be an interesting challenge to understand how such a coupling can be realized.

Ultimately, the entire physics on H should be dual to a $2 + 1$ dimensional CFT on ∂Y. This reduces the holographic dual of the global bulk space-time to a CFT on Y with boundary dynamics at ∂Y. This would follow from the conjectured duality in ref. [15] which has been partially verified in ref. [32]. In this picture, the dynamics we have found would take a particularly striking form. All physics in $\mathcal{D}(H)$ would be dual (via H) to data on the boundary ∂Y of the CFT on Y. Excitations on Y typically reach ∂Y and get stuck there for a time equal to the light-crossing time of Y. Hence, the boundary dynamics at ∂Y must be highly non-trivial.

In ref. [14] it was noted that four-dimensional gravity should break down at sufficiently large scales on the AdS_4 brane. This does not happen in the case of the RS brane, where 4D gravity is valid at all scales. The difference can be ascribed to the divergence of the warp factor in the bulk region X. But in section 6.2 we showed that an experiment of sufficient duration to probe the entire brane is just at the threshold to probing X. This suggests that four-dimensional gravity does not break down at finite *distances* on the AdS_4 brane. Rather, the relevant scale is a time scale. An experiment longer than $\pi\ell_4$ will detect the growing warp factor beyond $r = 0$, and hence will see a region where gravity appears genuinely five-dimensional. It is significant that X also coincides with the bulk region outside the brane's holographic domain, as we discuss in the final section.

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8. Interpretation

In both the RS and the KR models, the holographic domain of the full boundary is obviously the entire bulk space. Four-dimensional gravity localizes near the brane, and bulk physics is expected to be dual to CFT excitations on the brane. In the KR model, however, the boundary consists of two distinct portions: the AdS_4 brane at $r = c$, and the half-AdS_5 boundary at $r = -\infty$.

In order to interpret the regime that exhibits four-dimensional gravity, it is necessary to separately identify the holographic domain associated with each portion of the boundary. A naive, space-like relation between brane and bulk data cannot distinguish domains and would violate the holographic entropy bound. However, we were able to obtain distinct domains by using the light-sheet construction central to the covariant formulation of the holographic principle. We found that the domain of the brane extends only to the minimum of the warp factor, at $r = 0$. The remainder of the bulk, $r < 0$, coincides with the holographic domain of the half-AdS_5 boundary.

We developed a technique based on causality that allowed us to locate the holographic dual on the boundary as a function of the position of a bulk excitation. In the RS model, we found that the causal diamond method reproduces previous results obtained by more intricate means. In the KR model, we were able to perform an important consistency check. The holographic image of a bulk excitation, as constructed via causal diamonds, respects our association of holographic domains to the two portions of the boundary. The image always lies on the portion of the boundary in whose holographic domain the bulk excitation is localized.

Generic physical processes, such as freely falling particles, thus correspond to highly nontrivial dynamics on the boundary. From the global perspective, we see that bulk excitations can cross from one holographic domain to the other, corresponding to conformal field theory excitations interacting over the "boundary of the boundary", ∂H.

An alternative description takes the point of view of the observer on the brane. Experiments longer than ℓ and shorter than ℓ_4 will measure four-dimensional gravitational effects. Experiments of greater duration are able to probe X, the bulk region outside the brane's holographic domain. Hence they are able to detect information that cannot be contained in the CFT on the brane. The brane observer can interpret this in two ways. One perspective is to note that such experiments probe beyond the minimum of the warp factor and see that gravity is five-dimensional at large time-scales. A second option is to retain the four-dimensional perspective: the observer lives in an AdS_4 space with non-standard boundary conditions, namely not reflective boundary conditions, but those determined by a CFT on Y. We can think of the AdS_4 brane as being glued across its conformal boundary to (the $\ell_4 \to \infty$ limit of) another AdS_4 space. Five-dimensional gravity is re-interpreted as dynamics in a conformal AdS_4 space-time adjacent to the AdS_4 brane. For example, the half-period part of an oscillating bulk particle's path that lies in X corresponds to its holographic image dipping into and out of that second (conformal) AdS_4 space.

Finally, one may also describe the entire AdS_4/AdS_5 system as a CFT on Y with non-trivial boundary dynamics on ∂Y. Such a theory would have to be able to retain

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excitations on the boundary for a long but finite period, corresponding to the time that bulk excitations spend within $\mathcal{D}(H)$.

With the discovery and detailed study of novel space-time backgrounds, we not only witness the emergence of four-dimensional gravity under ever more general conditions, but we also learn new lessons that might yield fundamental insights. In this paper, we have seen that the consistency of holography in the KR model is upheld by the covariant formulation of the holographic principle, providing, in turn, support for this formulation and new insights into its workings.

There is much left to be understood about gravity. The discovery of new geometries has already revealed characteristics previously thought to be inadmissible. For example, the existence [14, 33] and consistency [34, 35, 36] of a massive graviton was not anticipated but is now fairly well understood. In this paper, we have done a detailed study of another new and surprising aspect of the AdS_4/AdS_5 geometry, namely the coexistence of different holographic domains reflecting different dimensionality. It is now clear that new geometries can give rise to fascinating new phenomena which might well provide critical clues to addressing the nature of gravity.

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